Int J Solids Structures Vol 22, No 9, pp 1019–1032, 1986
Printed in Great Britain

0020–7683/86 $3.00+.00
Pergamon Journals Ltd

THE MATERIAL TIME DERIVATIVE OF LOGARITHMIC STRAIN

ANNE HOGER
Department of Mathematics, Carnegie-Mellon University, Pittsburgh, PA 15213, U.S.A.

(*Received* 3 *December* 1984; *in revised form* 1 *July* 1985)

Abstract—In this article we derive explicit formulas for the time rate of change of the logarithmic strains ln **U** and ln **V**, where **U** and **V** are the right and left stretch tensors, respectively. Results are obtained in both two and three dimensions for the cases where the principal stretches are repeated as well as for the case where they are distinct. The formulas are displayed in terms of **V**, the stretching tensor **D**, the rotation **R**, and the principal stretches. Finally, we consider the corotational and Jaumann derivatives of ln **V**, and establish conditions under which these logarithmic strain rates are equal to the stretching tensor.

1. INTRODUCTION

The logarithmic, or Hencky, strain has been considered to be a useful strain measure in one dimension, and also in two and three dimensions when the principal axes of strain are fixed. In these cases the stretching tensor **D** and the logarithm of the left stretch tensor **V** are related through

$$(\ln \mathbf{V})^{\cdot} = \mathbf{D}. \tag{1.1}$$

The use of logarithmic strain as a strain measure in the general case has been hampered by the lack of a properly invariant expression relating the time rate of change of the logarithmic strain to the stretching tensor. Hill[1] derived a relationship between the stretching tensor and the logarithm of the right stretch tensor **U** for the case where the principal stretches are distinct. His result is expressed in terms of the components of $(\ln \mathbf{U})^{\cdot}$ and **D** taken with respect to the principal axes of **U** and **V**, respectively. More recently, Gurtin and Spear[2] established necessary and sufficient conditions for the Jaumann derivative of ln **V** to be equal to **D** when the number of distinct principal stretches is constant.

In this article we derive explicit formulas for the time rate of change of the logarithmic strain tensors ln **U** and ln **V** in terms of the stretching tensor. The formulas, which are displayed in direct notation, are valid not only when the principal stretches are distinct, but also when they are repeated.

In Section 2 we briefly summarize the kinematical results that will be used in the remainder of the article. Two theorems that provide expressions for the derivative of a symmetric tensor valued function of a symmetric tensor are quoted in Section 3, and then applied to ln **U** in the next two sections. The method for obtaining $(\ln \mathbf{U})^{\cdot}$ is the same in both two and three dimensions, but the details of the derivation are different; the two-dimensional case is treated in Section 4 and the three-dimensional case in Section 5. In each case the expression derived for $(\ln \mathbf{U})^{\cdot}$ is displayed in terms of **V**, **D**, the principal stretches, and the rotation **R**. A simple relation between $(\ln \mathbf{U})^{\cdot}$ and $(\ln \mathbf{V})^{\cdot}$ is established, and used to obtain formulas for $(\ln \mathbf{V})^{\cdot}$. Our results are compared with those of Hill[1] at the end of Section 5. In the last section we consider the corotational and Jaumann derivatives of ln **V**, and establish that **DV** = **VD** is a necessary and sufficient condition for these logarithmic strain rates to be equal to the stretching tensor. On intervals during which the number of distinct principal stretches is constant our condition is equivalent to that obtained by Gurtin and Spear[2].

2. PRELIMINARIES†

Let $\mathbf{F}$ denote the deformation gradient at a point of a deforming body. Because $\det \mathbf{F} > 0$, we have the unique polar decompositions

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}. \tag{2.1}$$

$\mathbf{U}$ and $\mathbf{V}$, the right and left stretch tensors, respectively, are symmetric and positive definite, and $\mathbf{R}$, the rotation, is proper orthogonal.

The eigenvalues of $\mathbf{U}$ (which are also those of $\mathbf{V}$) are the principal stretches. They will be denoted by λ_i, $i = 1, 2$ for two dimensions and $i = 1, 2, 3$ for three.‡ One can choose an orthonormal basis of eigenvectors for $\mathbf{U}$; let $\{\mathbf{e}_i\}$ constitute such a basis, where $\mathbf{e}_i$ corresponds to λ_i.

By the spectral theorem, the right and left stretch tensors admit the representations

$$\mathbf{U} = \sum_{i=1}^{n} \lambda_i \mathbf{e}_i \otimes \mathbf{e}_i \tag{2.2§}$$

and

$$\mathbf{V} = \sum_{i=1}^{n} \lambda_i \bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}_i, \tag{2.3}$$

respectively. The eigenvectors of $\mathbf{V}$, $\bar{\mathbf{e}}_i$, are related to the eigenvectors of $\mathbf{U}$ through

$$\bar{\mathbf{e}}_i = \mathbf{R}\mathbf{e}_i. \tag{2.4}$$

Note that

$$\sum_{i=1}^{n} \mathbf{e}_i \otimes \mathbf{e}_i = \sum_{i=1}^{n} \bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}_i = \mathbf{1}. \tag{2.5}$$

The tensor logarithm maps positive definite, symmetric tensors into symmetric tensors. The logarithmic tensors are defined by

$$\ln \mathbf{U} = \sum_{i=1}^{n} \ln \lambda_i \mathbf{e}_i \otimes \mathbf{e}_i \tag{2.6}$$

and

$$\ln \mathbf{V} = \sum_{i=1}^{n} \ln \lambda_i \bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}_i. \tag{2.7}$$

In view of (2.4),

$$\ln \mathbf{U} = \mathbf{R}^T(\ln \mathbf{V})\mathbf{R}. \tag{2.8}$$

We assume that $\mathbf{F}$ is a continuously differentiable function of time. The velocity gradient

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} \tag{2.9}$$

has as its symmetric part the stretching tensor

$$\mathbf{D} = \tfrac{1}{2}(\mathbf{L}+\mathbf{L}^T) \tag{2.10}$$

† A complete treatment of the material in this section can be found in [3].
‡ In what follows, the dimension of the underlying space is denoted by n, with $n = 2$ or 3.
§ Throughout the article, no summation is implied unless explicitly indicated.

and as its skew part the spin tensor

$$\mathbf{W} = \tfrac{1}{2}(\mathbf{L}-\mathbf{L}^T). \tag{2.11}$$

By combining (2.9) and (2.10), we find that $\dot{\mathbf{U}}$ and $\mathbf{D}$ are related through

$$2\mathbf{R}^T\mathbf{D}\mathbf{R} = \dot{\mathbf{U}}\mathbf{U}^{-1}+\mathbf{U}^{-1}\dot{\mathbf{U}}. \tag{2.12}$$

Pre- and post-multiplication of (2.12) by $\mathbf{U}$ yields

$$2\mathbf{F}^T\mathbf{D}\mathbf{F} = \mathbf{U}\dot{\mathbf{U}}+\dot{\mathbf{U}}\mathbf{U}. \tag{2.13}$$

Substitution of (2.9) into (2.11) implies

$$\mathbf{W} = \dot{\mathbf{R}}\mathbf{R}^T+\tfrac{1}{2}\mathbf{R}(\dot{\mathbf{U}}\mathbf{U}^{-1}-\mathbf{U}^{-1}\dot{\mathbf{U}})\mathbf{R}^T;$$

thus, with

$$\mathbf{\Omega} = \dot{\mathbf{R}}\mathbf{R}^T, \tag{2.14}$$

$$\mathbf{W} = \mathbf{\Omega}+\mathbf{D}-\mathbf{R}\mathbf{U}^{-1}\dot{\mathbf{U}}\mathbf{R}^T. \tag{2.15}$$

Once an expression for $(\ln \mathbf{U})^{\cdot}$ has been derived, the formula for $(\ln \mathbf{V})^{\cdot}$ can easily be obtained with the aid of the following result. Differentiation of (2.8) yields

$$(\ln \mathbf{U})^{\cdot} = \dot{\mathbf{R}}^T(\ln \mathbf{V})\mathbf{R}+\mathbf{R}^T(\ln \mathbf{V})^{\cdot}\mathbf{R}+\mathbf{R}^T(\ln \mathbf{V})\dot{\mathbf{R}},$$

so, because $\mathbf{\Omega}$ is skew,

$$(\ln \mathbf{V})^{\cdot} = \mathbf{R}(\ln \mathbf{U})^{\cdot}\mathbf{R}^T+\mathbf{\Omega}(\ln \mathbf{V})-(\ln \mathbf{V})\mathbf{\Omega}. \tag{2.16}$$

Finally, we recall the definitions of two objective time rates of change. Let $\mathbf{A}$ be a smooth tensor function of time. The corotational derivative of $\mathbf{A}$, denoted by $\mathring{\mathbf{A}}$, is given by

$$\mathring{\mathbf{A}} = \dot{\mathbf{A}}+\mathbf{A}\mathbf{\Omega}-\mathbf{\Omega}\mathbf{A}.$$

The Jaumann rate $\overset{*}{\mathbf{A}}$ is defined as

$$\overset{*}{\mathbf{A}} = \dot{\mathbf{A}}+\mathbf{A}\mathbf{W}-\mathbf{W}\mathbf{A}.$$

3. THE DERIVATIVE OF SPECIAL TENSOR VALUED FUNCTIONS OF A TENSOR

Let $\mathbf{H}$ be a symmetric tensor valued function of a symmetric tensor. $\mathbf{H}$ is said to be differentiable at $\mathbf{X}$ if there exists a linear transformation $D\mathbf{H}(\mathbf{X})$ such that

$$\mathbf{H}(\mathbf{X}+\mathbf{T}) = \mathbf{H}(\mathbf{X})+D\mathbf{H}(\mathbf{X})[\mathbf{T}]+o(\mathbf{T})$$

as $\mathbf{T}\to\mathbf{0}$. $D\mathbf{H}(\mathbf{X})$ is the derivative of $\mathbf{H}$ at $\mathbf{X}$, and the symmetric tensor $\mathbf{T}$ is the increment on which the derivative is evaluated. If $\mathbf{X}$ is a differentiable function of time, then the chain rule states that

$$[\mathbf{H}(\mathbf{X}(t))]^{\cdot} = D\mathbf{H}(\mathbf{X})[\dot{\mathbf{X}}(t)]. \tag{3.1}$$

In order to implement the chain rule we need an expression for $D\mathbf{H}(\mathbf{X})[\mathbf{T}]$. The following two theorems, which were established in [4], provide the required formulas.

Theorem ($n = 2$). *Let* $\mathbf{X}$ *be a symmetric tensor with eigenvalues* x_1, x_2 *and an associated orthonormal eigenvector basis* $\{\mathbf{e}_1, \mathbf{e}_2\}$. *Given the real valued function* h, *define the tensor* *valued function* $\mathbf{H}$ *by*

$$\mathbf{H}(\mathbf{X}) = h(x_1)\mathbf{e}_1 \otimes \mathbf{e}_1 + h(x_2)\mathbf{e}_2 \otimes \mathbf{e}_2.$$

Then, if h is four times continuously differentiable, $\mathbf{H}$ *is continuously differentiable, and*

$$\begin{aligned} D\mathbf{H}(\mathbf{X})[\mathbf{T}] = \frac{1}{(x_1-x_2)^3}\,[\{(x_1-x_2)\,[h'(x_1)+h'(x_2)]-2[h(x_1)-h(x_2)]\}\mathbf{XTX} \\ +\{-(x_1-x_2)\,[x_2h'(x_1)+x_1h'(x_2)]+(x_1+x_2)\,[h(x_1)-h(x_2)]\}\,(\mathbf{XT}+\mathbf{TX}) \\ +\{(x_1-x_2)\,[x_2^2h'(x_1)+x_1^2h'(x_2)]-2x_1x_2[h(x_1)-h(x_2)]\}\mathbf{T}], \qquad x_1 \neq x_2, \end{aligned} \tag{3.2}$$

$$D\mathbf{H}(\mathbf{X})[\mathbf{T}] = h'(x)\mathbf{T}, \qquad x_1 = x_2 \equiv x. \tag{3.3}$$

Here and in the following theorem a prime denotes ordinary differentiation.

Theorem ($n = 3$). *Let* $\mathbf{X}$ *be a symmetric tensor with eigenvalues* x_1, x_2, x_3 *and an associated orthonormal eigenvector basis* $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. *Given the real valued function h, define the tensor valued function* $\mathbf{H}$ *by*

$$\mathbf{H}(\mathbf{X}) = h(x_1)\mathbf{e}_1 \otimes \mathbf{e}_1 + h(x_2)\mathbf{e}_2 \otimes \mathbf{e}_2 + h(x_3)\mathbf{e}_3 \otimes \mathbf{e}_3.$$

Then, if h is seven times continuously differentiable, $\mathbf{H}$ *is continuously differentiable, and*

$$\begin{aligned} D\mathbf{H}(\mathbf{X})[\mathbf{T}] = (\mathscr{F}_1+\mathscr{F}_2+\mathscr{F}_3)\mathbf{X}^2\mathbf{T}\mathbf{X}^2 \\ -[(x_1+x_2)\mathscr{F}_3+(x_2+x_3)\mathscr{F}_1+(x_3+x_1)\mathscr{F}_2]\,(\mathbf{X}^2\mathbf{TX}+\mathbf{XTX}^2) \\ +(x_1x_2\mathscr{F}_3+x_2x_3\mathscr{F}_1+x_3x_1\mathscr{F}_2)\,(\mathbf{X}^2\mathbf{T}+\mathbf{TX}^2) \\ +[(x_1+x_2)^2\mathscr{F}_3+(x_2+x_3)^2\mathscr{F}_1+(x_3+x_1)^2\mathscr{F}_2]\mathbf{XTX} \\ +[-x_1x_2(x_1+x_2)\mathscr{F}_3-x_2x_3(x_2+x_3)\mathscr{F}_1-x_3x_1(x_3+x_1)\mathscr{F}_2 \\ +\mathscr{G}_1+\mathscr{G}_2+\mathscr{G}_3]\,(\mathbf{XT}+\mathbf{TX})+[(x_1x_2)^2\mathscr{F}_3+(x_2x_3)^2\mathscr{F}_1+(x_3x_1)^2\mathscr{F}_2 \\ -(x_1+x_2)\mathscr{G}_3-(x_2+x_3)\mathscr{G}_1-(x_3+x_1)\mathscr{G}_2]\mathbf{T}, \qquad x_1 \neq x_2 \neq x_3 \neq x_1, \end{aligned} \tag{3.4}$$

$$\begin{aligned} D\mathbf{H}(\mathbf{X})[\mathbf{T}] = \frac{1}{(x_i-x)^3}\,[\{(x_i-x)\,[h'(x_i)+h'(x)]-2[h(x_i)-h(x)]\}\mathbf{XTX} \\ +\{-(x_i-x)\,[xh'(x_i)+x_ih'(x)]+(x_i+x)\,[h(x_i)-h(x)]\}\,(\mathbf{XT}+\mathbf{TX}) \\ +\{(x_i-x)\,[x^2h'(x_i)+x_i^2h'(x)]-2x_ix[h(x_i)-h(x)]\}\mathbf{T}], \qquad x_i \neq x_j = x_k \equiv x, \end{aligned} \tag{3.5}$$

$$D\mathbf{H}(\mathbf{X})[\mathbf{T}] = h'(x)\mathbf{T}, \qquad x_1 = x_2 = x_3 \equiv x, \tag{3.6}$$

where

$$\mathscr{G}_i = \frac{h(x_i)}{(x_i-x_j)\,(x_i-x_k)},$$

$$\mathscr{F}_i = \frac{1}{(x_i-x_j)^2(x_i-x_k)^2}[h'(x_i)-(x_i-x_j)\,(\mathscr{G}_i+\mathscr{G}_k)-(x_i-x_k)\,(\mathscr{G}_i+\mathscr{G}_j)],$$

and i, j, k *is a permutation of* 1, 2, 3 (*no summation*).

Although the particular formula to be used for the derivative depends on the number of distinct eigenvalues of $\mathbf{X}$, it was shown in [4] that $D\mathbf{H}(\mathbf{X})[\mathbf{T}]$ is a continuous function of $\mathbf{X}$.

4. TWO DIMENSIONS

The choices $h = \ln$, $\mathbf{X} = \mathbf{U}$ and $\mathbf{T} = \dot{\mathbf{U}}$ in (3.2) and (3.3) and use of (3.1) yield the following expression for the time rate of change of ln **U** in two dimensions:

$$
(\ln \mathbf{U})^{\cdot} = \begin{cases} \left(\dfrac{1}{\lambda_1-\lambda_2}\right)^3 \left\{ \left[\left(\dfrac{1}{\lambda_1}+\dfrac{1}{\lambda_2}\right)(\lambda_1-\lambda_2) - 2\ln(\lambda_1/\lambda_2)\right]\mathbf{U}\dot{\mathbf{U}}\mathbf{U} \right. \\ \quad + \left[-\left(\dfrac{\lambda_2}{\lambda_1}+\dfrac{\lambda_1}{\lambda_2}\right)(\lambda_1-\lambda_2)+(\lambda_1+\lambda_2)\ln(\lambda_1/\lambda_2)\right](\mathbf{U}\dot{\mathbf{U}}+\dot{\mathbf{U}}\mathbf{U}) \\ \quad \left. + \left[\left(\dfrac{\lambda_2^2}{\lambda_1}+\dfrac{\lambda_1^2}{\lambda_2}\right)(\lambda_1-\lambda_2) - 2\lambda_1\lambda_2 \ln(\lambda_1/\lambda_2)\right]\dot{\mathbf{U}}\right\}, \quad \lambda_1 \neq \lambda_2, & (4.1) \\ \dfrac{1}{\lambda}\dot{\mathbf{U}}, \quad \lambda_1 = \lambda_2 \equiv \lambda. & (4.2) \end{cases}
$$

The reader will recall that λ_1 and λ_2 are the principal stretches.

In order to write (ln **U**)˙ directly in terms of the stretching tensor, the tensor terms on the right hand side of these equations must be expressed in terms of **D**. Equation (2.12),

$$\dot{\mathbf{U}}\mathbf{U}^{-1}+\mathbf{U}^{-1}\dot{\mathbf{U}} = 2\mathbf{R}^T\mathbf{D}\mathbf{R},$$

can be solved for $\dot{\mathbf{U}}$ with the aid of formulas obtained in [5, §2]. The result in two dimensions is

$$\dot{\mathbf{U}} = \frac{1}{\mathrm{I}}\,[\mathrm{II}^2\mathbf{U}^{-1}\mathbf{R}^T\mathbf{D}\mathbf{R}\mathbf{U}^{-1} - \mathrm{I}\ \mathrm{II}(\mathbf{U}^{-1}\mathbf{R}^T\mathbf{D}\mathbf{R}+\mathbf{R}^T\mathbf{D}\mathbf{R}\mathbf{U}^{-1})+(\mathrm{I}^2+\mathrm{II})\mathbf{R}^T\mathbf{D}\mathbf{R}], \tag{4.3}$$

where I and II are the principal invariants of **U**:

$$\mathrm{I} = \lambda_1+\lambda_2,$$
$$\mathrm{II} = \lambda_1\lambda_2.$$

The Cayley–Hamilton theorem for **U** yields

$$\mathbf{U}^{-1} = \frac{1}{\mathrm{II}}\,[-\mathbf{U}+\mathrm{I}\mathbf{1}], \tag{4.4}$$

so that (4.3) can be expressed as

$$\dot{\mathbf{U}} = \frac{1}{\mathrm{I}}\,[\mathrm{II}\mathbf{R}^T\mathbf{D}\mathbf{R}+\mathbf{F}^T\mathbf{D}\mathbf{F}]. \tag{4.5}$$

Equation (4.3) implies

$$\mathbf{U}\dot{\mathbf{U}}\mathbf{U} = \frac{1}{\mathrm{I}}\,[\mathrm{II}^2\mathbf{R}^T\mathbf{D}\mathbf{R} - \mathrm{I}\ \mathrm{II}(\mathbf{R}^T\mathbf{D}\mathbf{F}+\mathbf{F}^T\mathbf{D}\mathbf{R})+(\mathrm{I}^2+\mathrm{II})\mathbf{F}^T\mathbf{D}\mathbf{F}]. \tag{4.6}$$

The remaining term, $\mathbf{U}\dot{\mathbf{U}}+\dot{\mathbf{U}}\mathbf{U}$, is given in terms of **D** by (2.13):

$$\mathbf{U}\dot{\mathbf{U}}+\dot{\mathbf{U}}\mathbf{U} = 2\mathbf{F}^T\mathbf{D}\mathbf{F}. \tag{4.7}$$

Substitution of (4.5)–(4.7) into (4.1) and use of (2.1) gives

$$(\ln \mathbf{U})^{\cdot} = \mathbf{R}^T[\Phi_1\mathbf{D}+\Phi_2(\mathbf{V}\mathbf{D}+\mathbf{D}\mathbf{V})+\Phi_3\mathbf{V}\mathbf{D}\mathbf{V}]\mathbf{R}, \qquad \lambda_1 \neq \lambda_2, \tag{4.8}$$

where

$$
\begin{aligned}
\Phi_1 &= \frac{\lambda_1^4-\lambda_2^4-4\lambda_1^2\lambda_2^2\ln(\lambda_1/\lambda_2)}{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)^3},\\
\Phi_2 &= \frac{-(\lambda_1^2-\lambda_2^2)+2\lambda_1\lambda_2\ln(\lambda_1/\lambda_2)}{(\lambda_1-\lambda_2)^3},\\
\Phi_3 &= \frac{2(\lambda_1^2-\lambda_2^2)-4\lambda_1\lambda_2\ln(\lambda_1/\lambda_2)}{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)^3}.
\end{aligned} \tag{4.9}
$$

When the principal stretches are equal ($\lambda_1 = \lambda_2 \equiv \lambda$), $\mathbf{U}$ is isotropic, so that (2.12) collapses to

$$\dot{\mathbf{U}} = \lambda \mathbf{R}^T\mathbf{D}\mathbf{R},$$

and (4.2) can be written as

$$(\ln \mathbf{U})^{\cdot} = \mathbf{R}^T\mathbf{D}\mathbf{R}, \qquad \lambda_1 = \lambda_2. \tag{4.10}$$

We have established the following result. *In two dimensions* $(\ln \mathbf{U})^{\cdot}$ *is given in terms of the stretching tensor* $\mathbf{D}$ *by* (4.8) *when the principal stretches are distinct and by* (4.10) *when the principal stretches coalesce.*

A formula for $(\ln \mathbf{V})^{\cdot}$ can now be obtained by use of (2.16). When specialized to two dimensions (2.3) and (2.7) together with (2.5) imply

$$\mathbf{V} = \lambda_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2(\mathbf{1} - \mathbf{e}_1 \otimes \mathbf{e}_1) \tag{4.11}$$

and

$$\ln \mathbf{V} = (\ln \lambda_1)\mathbf{e}_1 \otimes \mathbf{e}_1 + (\ln \lambda_2)(\mathbf{1} - \mathbf{e}_1 \otimes \mathbf{e}_1), \tag{4.12}$$

respectively. Suppose that $\lambda_1 \neq \lambda_2$; then (4.11) gives

$$\mathbf{e}_1 \otimes \mathbf{e}_1 = \frac{\mathbf{V} - \lambda_2 \mathbf{1}}{\lambda_1 - \lambda_2},$$

which can be employed in (4.12) with the result

$$\ln \mathbf{V} = \frac{\ln(\lambda_1/\lambda_2)}{\lambda_1 - \lambda_2}\mathbf{V} + \frac{\lambda_1 \ln \lambda_2 - \lambda_2 \ln \lambda_1}{\lambda_1 - \lambda_2}\mathbf{1}. \tag{4.13}$$

Clearly

$$\mathbf{\Omega}(\ln \mathbf{V}) - (\ln \mathbf{V})\mathbf{\Omega} = \frac{\ln(\lambda_1/\lambda_2)}{\lambda_1 - \lambda_2}(\mathbf{\Omega}\mathbf{V} - \mathbf{V}\mathbf{\Omega}). \tag{4.14}$$

Suppose that $\lambda_1 = \lambda_2$; in this case $\ln \mathbf{V}$ is isotropic so

$$\mathbf{\Omega}(\ln \mathbf{V}) - (\ln \mathbf{V})\mathbf{\Omega} = \mathbf{0}. \tag{4.15}$$

Equations (2.16), (4.8), (4.10), (4.14), and (4.15) may be used to establish that *in two dimensions* $(\ln \mathbf{V})^{\cdot}$ *is given in terms of the stretching tensor* $\mathbf{D}$ *and* $\mathbf{\Omega} = \dot{\mathbf{R}}\mathbf{R}^T$ *by*

$$(\ln \mathbf{V})^{\cdot} = \begin{cases} \Phi_1 \mathbf{D} + \Phi_2(\mathbf{V}\mathbf{D} + \mathbf{D}\mathbf{V}) + \Phi_3 \mathbf{V}\mathbf{D}\mathbf{V} + \dfrac{\ln \lambda_1/\lambda_2}{(\lambda_1 - \lambda_2)}(\mathbf{\Omega}\mathbf{V} - \mathbf{V}\mathbf{\Omega}), & \lambda_1 \neq \lambda_2, \\ \mathbf{D}, \quad \lambda_1 = \lambda_2, \end{cases} \tag{4.16}$$

where Φ_1, Φ_2, *and* Φ_3 *are defined through* (4.9).

Finally, we note that with $\mathbf{U}^{-1}$ represented by (4.4) and $\dot{\mathbf{U}}$ represented by (4.5), (2.15) yields

$$\mathbf{\Omega} = \mathbf{W} + \frac{1}{\lambda_1 + \lambda_2}(\mathbf{D}\mathbf{V} - \mathbf{V}\mathbf{D}). \tag{4.17}$$

Thus an expression for $(\ln \mathbf{V})^{\cdot}$ in terms of $\mathbf{W}$ rather than $\mathbf{\Omega}$ can be obtained by substitution of (4.17) into (4.16).

5. THREE DIMENSIONS

The calculations necessary for the derivation of $(\ln \mathbf{U})^{\cdot}$ in terms of $\mathbf{D}$ for the three-dimensional case are similar to but more cumbersome than those performed for the two-dimensional case. By choosing $h = \ln$, $\mathbf{X} = \mathbf{U}$ and $\mathbf{T} = \dot{\mathbf{U}}$ in (3.4)–(3.6) and noting (3.1), we obtain the following formula for the time rate of change of $\ln \mathbf{U}$ in three dimensions:

$$
(\ln \mathbf{U})^{\cdot} = \begin{cases}
[\mathscr{F}_1+\mathscr{F}_2+\mathscr{F}_3]\mathbf{U}^2\dot{\mathbf{U}}\mathbf{U}^2 \\
-[(\lambda_1+\lambda_2)\mathscr{F}_3+(\lambda_2+\lambda_3)\mathscr{F}_1+(\lambda_3+\lambda_1)\mathscr{F}_2]\,(\mathbf{U}^2\dot{\mathbf{U}}\mathbf{U}+\mathbf{U}\dot{\mathbf{U}}\mathbf{U}^2) \\
+[\lambda_1\lambda_2\mathscr{F}_3+\lambda_2\lambda_3\mathscr{F}_1+\lambda_3\lambda_1\mathscr{F}_2](\mathbf{U}^2\dot{\mathbf{U}}+\dot{\mathbf{U}}\mathbf{U}^2) \\
+[(\lambda_1+\lambda_2)^2\mathscr{F}_3+(\lambda_2+\lambda_3)^2\mathscr{F}_1+(\lambda_3+\lambda_1)^2\mathscr{F}_2]\mathbf{U}\dot{\mathbf{U}}\mathbf{U} \\
-[\lambda_1\lambda_2(\lambda_1+\lambda_2)\mathscr{F}_3+\lambda_2\lambda_3(\lambda_2+\lambda_3)\mathscr{F}_1+\lambda_3\lambda_1(\lambda_3+\lambda_1)\mathscr{F}_2 \\
-\mathscr{G}_1-\mathscr{G}_2-\mathscr{G}_3]\,(\mathbf{U}\dot{\mathbf{U}}+\dot{\mathbf{U}}\mathbf{U}) \\
+[\lambda_1^2\lambda_2^2\mathscr{F}_3+\lambda_2^2\lambda_3^2\mathscr{F}_1+\lambda_3^2\lambda_1^2\mathscr{F}_2-(\lambda_1+\lambda_2)\mathscr{G}_3 \\
-(\lambda_2+\lambda_3)\mathscr{G}_1-(\lambda_3+\lambda_1)\mathscr{G}_2]\dot{\mathbf{U}}, \qquad \lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1, & (5.1) \\
\left(\dfrac{1}{\lambda_i-\lambda}\right)^3\left\{\left[\left(\dfrac{1}{\lambda_i}+\dfrac{1}{\lambda}\right)(\lambda_i-\lambda)-2\ln(\lambda_i/\lambda)\right]\mathbf{U}\dot{\mathbf{U}}\mathbf{U}\right. \\
+\left[-\left(\dfrac{\lambda}{\lambda_i}+\dfrac{\lambda_i}{\lambda}\right)(\lambda_i-\lambda)+(\lambda_i+\lambda)\ln(\lambda_i/\lambda)\right](\mathbf{U}\dot{\mathbf{U}}+\dot{\mathbf{U}}\mathbf{U}) & (5.2) \\
\left.+\left[\left(\dfrac{\lambda^2}{\lambda_i}+\dfrac{\lambda_i^2}{\lambda}\right)(\lambda_i-\lambda)-2\lambda_i\lambda\ln(\lambda_i/\lambda)\right]\dot{\mathbf{U}}\right\}, \qquad \lambda_i \neq \lambda_j = \lambda_k \equiv \lambda, \\
\dfrac{1}{\lambda}\dot{\mathbf{U}}, \qquad \lambda_1 = \lambda_2 = \lambda_3 \equiv \lambda, & (5.3)
\end{cases}
$$

where

$$
\begin{aligned}
\mathscr{G}_i &= \frac{\ln \lambda_i}{(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)}, \\
\mathscr{F}_i &= \left[\frac{1}{(\lambda_i-\lambda_j)^2(\lambda_i-\lambda_k)^2}\right]\left[\frac{1}{\lambda_i}-(\lambda_i-\lambda_j)(\mathscr{G}_i+\mathscr{G}_k)-(\lambda_i-\lambda_k)(\mathscr{G}_i+\mathscr{G}_j)\right],
\end{aligned}
\tag{5.4}
$$

and i, j, k is a permutation of 1, 2, 3 (no summation).

The manipulations that are required to write the tensor terms in each of the above equations, and hence $(\ln \mathbf{U})^{\cdot}$, in terms of $\mathbf{D}$ rather than $\dot{\mathbf{U}}$ again rest on the solution of eqn (2.13) [or alternatively eqn (2.12)] for $\dot{\mathbf{U}}$. Using a solution formula derived in [5, §2] (see also [6]), we find that in three dimensions

$$
\begin{aligned}
\dot{\mathbf{U}} = \frac{1}{(\mathrm{I}\,\mathrm{II}-\mathrm{III})\mathrm{III}}\Big\{&\mathrm{I}\mathbf{U}^2\mathbf{F}^T\mathbf{D}\mathbf{F}\mathbf{U}^2-\mathrm{I}^2(\mathbf{U}^2\mathbf{F}^T\mathbf{D}\mathbf{F}\mathbf{U}+\mathbf{U}\mathbf{F}^T\mathbf{D}\mathbf{F}\mathbf{U}^2) \\
&+(\mathrm{I}\,\mathrm{II}-\mathrm{III})(\mathbf{U}^2\mathbf{F}^T\mathbf{D}\mathbf{F}+\mathbf{F}^T\mathbf{D}\mathbf{F}\mathbf{U}^2)+(\mathrm{I}^3+\mathrm{III})\mathbf{U}\mathbf{F}^T\mathbf{D}\mathbf{F}\mathbf{U} \\
&-\mathrm{I}^2\mathrm{II}(\mathbf{U}\mathbf{F}^T\mathbf{D}\mathbf{F}+\mathbf{F}^T\mathbf{D}\mathbf{F}\mathbf{U})+[\mathrm{I}^2\mathrm{III}+(\mathrm{I}\,\mathrm{II}-\mathrm{III})\mathrm{II}]\mathbf{F}^T\mathbf{D}\mathbf{F}\Big\},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathrm{I} &= \lambda_1+\lambda_2+\lambda_3, \\
\mathrm{II} &= \lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1, \\
\mathrm{III} &= \lambda_1\lambda_2\lambda_3
\end{aligned}
$$

are the principal invariants of $\mathbf{U}$. The expression for $\dot{\mathbf{U}}$ can be written more compactly with the aid of (2.1) and the Cayley–Hamilton theorem for $\mathbf{U}$,

$$\mathbf{U}^3-\mathrm{I}\,\mathbf{U}^2+\mathrm{II}\,\mathbf{U}-\mathrm{III}\,\mathbf{1}=\mathbf{0}. \tag{5.5}$$

We obtain

$$\dot{\mathbf{U}}=\frac{1}{(\mathrm{I\,II}-\mathrm{III})}\mathbf{R}^T\{\mathbf{V}^2\mathbf{D}\mathbf{V}^2-\mathrm{I}(\mathbf{V}^2\mathbf{D}\mathbf{V}+\mathbf{V}\mathbf{D}\mathbf{V}^2)+(\mathrm{I}^2+\mathrm{II})\mathbf{V}\mathbf{D}\mathbf{V}-\mathrm{III}(\mathbf{V}\mathbf{D}+\mathbf{D}\mathbf{V})+\mathrm{I\,III}\,\mathbf{D}\}\mathbf{R}. \tag{5.6}$$

Substitution of (5.6) into (5.1) followed by repeated application of (5.5) leads to an expression for $(\ln\mathbf{U})^{\cdot}$ in terms of the stretching tensor which is valid when the three principal stretches are distinct. The result is

$$\begin{aligned}(\ln\mathbf{U})^{\cdot}=\frac{1}{(\mathrm{I\,II}-\mathrm{III})}\mathbf{R}^T\{&\Lambda_1\mathbf{D}+\Lambda_2(\mathbf{D}\mathbf{V}+\mathbf{V}\mathbf{D})\\&+\Lambda_3(\mathbf{D}\mathbf{V}^2+\mathbf{V}^2\mathbf{D})\\&+\Lambda_4\mathbf{V}\mathbf{D}\mathbf{V}+\Lambda_5(\mathbf{V}^2\mathbf{D}\mathbf{V}+\mathbf{V}\mathbf{D}\mathbf{V}^2)+\Lambda_6\mathbf{V}^2\mathbf{D}\mathbf{V}^2\}\mathbf{R},\end{aligned} \tag{5.7}$$

where the Λ's are defined as follows. Recall that $\mathscr{F}_i$ and $\mathscr{G}_i$ are given by (5.4), and let

$$\varphi_i=\lambda_j\lambda_k,$$
$$\psi_i=\lambda_j+\lambda_k,$$

with i,j,k a permutation of $1,2,3$. Then

$$\begin{aligned}\Lambda_1&=\sum_{i=1}^{3}\mathrm{III}\{[\mathrm{II\,III}+(\psi_i^2-2\varphi_i)\mathrm{III}+\varphi_i^2\,\mathrm{I}]\mathscr{F}_i-\psi_i\mathrm{I}\mathscr{G}_i\},\\
\Lambda_2&=\sum_{i=1}^{3}\mathrm{III}\{[-\mathrm{II}^2-(\psi_i^2-2\varphi_i)\mathrm{II}-\varphi_i^2]\mathscr{F}_i+\psi_i\mathscr{G}_i\},\\
\Lambda_3&=\sum_{i=1}^{3}(\mathrm{I\,II}-\mathrm{III})\mathrm{III}\,\mathscr{F}_i,\\
\Lambda_4&=\sum_{i=1}^{3}\{[\mathrm{II}^3+\mathrm{III}^2+(\psi_i^2-2\varphi_i)(\mathrm{I\,III}+\mathrm{II}^2)\\
&\quad-2\psi_i\varphi_i(\mathrm{I\,II}-\mathrm{III})+\varphi_i^2(\mathrm{I}^2+\mathrm{II})]\mathscr{F}_i\\
&\quad+[2(\mathrm{I\,II}-\mathrm{III})-(\mathrm{I}^2+\mathrm{II})\psi_i]\mathscr{G}_i\},\\
\Lambda_5&=\sum_{i=1}^{3}\{[-\mathrm{I\,II}^2+2\varphi_i(\mathrm{I\,II}-\mathrm{III})-(\psi_i^2-2\varphi_i)\mathrm{III}-\varphi_i^2\mathrm{I}]\mathscr{F}_i+\mathrm{I}\psi_i\mathscr{G}_i\},\\
\Lambda_6&=\sum_{i=1}^{3}\{(\mathrm{I\,II}-\mathrm{III})\mathrm{I}+\mathrm{II}^2-2\psi_i(\mathrm{I\,II}-\mathrm{III})\\
&\quad+(\psi_i^2-2\varphi_\kappa)\mathrm{II}+\varphi_i^2]\mathscr{F}_i-\psi_i\mathscr{G}_i\}.\end{aligned} \tag{5.8}$$

Next, we consider the case where exactly two of the three principal stretches have the same value. Suppose that $\lambda_i\neq\lambda_j=\lambda_k\equiv\lambda$; then (2.3) and (2.5) imply

$$\mathbf{V}^2=(\lambda_i+\lambda)\mathbf{V}-\lambda_i\lambda\mathbf{1}. \tag{5.9}$$

Substitution of (5.6) and (5.9) into (5.2) yields

$$(\ln\mathbf{U})^{\cdot}=\mathbf{R}^T\{\Theta_1\mathbf{D}+\Theta_2(\mathbf{V}\mathbf{D}+\mathbf{D}\mathbf{V})+\Theta_3\mathbf{V}\mathbf{D}\mathbf{V}\}\mathbf{R},\qquad\lambda_i\neq\lambda_j=\lambda_k\equiv\lambda, \tag{5.10}$$

where

$$\Theta_1 = \frac{\lambda_i^4-\lambda^4-4\lambda_i^2\lambda^2\ln(\lambda_i/\lambda)}{(\lambda_i+\lambda)(\lambda_i-\lambda)^3},$$
$$\Theta_2 = -\frac{(\lambda_i^2-\lambda^2)+2\lambda_i\lambda\ln(\lambda_i/\lambda)}{(\lambda_i-\lambda)^3}, \tag{5.11}$$
$$\Theta_3 = \frac{2(\lambda_i^2-\lambda^2)-4\lambda_i\lambda\ln(\lambda_i/\lambda)}{(\lambda_i+\lambda)(\lambda_i-\lambda)^3},$$

and i,j,k is a permutation of 1, 2, 3. Note that this is essentially the same expression that was obtained in two dimensions for two distinct eigenvalues.

When the principal stretches are all equal ($\lambda_1 = \lambda_2 = \lambda_3 \equiv \lambda$), (2.12) becomes

$$\dot{\mathbf{U}} = \lambda\mathbf{R}^T\mathbf{D}\mathbf{R},$$

so that (5.3) is

$$(\ln \mathbf{U})^{\cdot} = \mathbf{R}^T\mathbf{D}\mathbf{R}, \qquad \lambda_1 = \lambda_2 = \lambda_3. \tag{5.12}$$

The results established above can be collected together as follows. *In three dimensions,* $(\ln \mathbf{U})^{\cdot}$ *is given in terms of the stretching tensor* $\mathbf{D}$ *by* (5.7) *when the principal stretches are all distinct, by* (5.10) *when exactly two principal stretches have the same value, and by* (5.12) *when all of the principal stretches coalesce.*

We now turn our attention to $\ln \mathbf{V}$. When specialized to three dimensions, (2.3), (2.5) and (2.7) can be used to calculate that

$$\mathbf{\Omega}(\ln \mathbf{V}) - (\ln \mathbf{V})\mathbf{\Omega} = \begin{cases} \Psi_1(\mathbf{\Omega}\mathbf{V}^2-\mathbf{V}^2\mathbf{\Omega})+\Psi_2(\mathbf{\Omega}\mathbf{V}-\mathbf{V}\mathbf{\Omega}), & \lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1, \\ \Psi_3(\mathbf{\Omega}\mathbf{V}-\mathbf{V}\mathbf{\Omega}), & \lambda_i \neq \lambda_j = \lambda_k \equiv \lambda, \\ \mathbf{0}, & \lambda_1 = \lambda_2 = \lambda_3, \end{cases} \tag{5.13}$$

where

$$\Psi_1 = \frac{\lambda_1\ln(\lambda_2/\lambda_3)+\lambda_2\ln(\lambda_3/\lambda_1)+\lambda_3\ln(\lambda_1/\lambda_2)}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)},$$
$$\Psi_2 = \frac{\lambda_1^2\ln(\lambda_3/\lambda_2)+\lambda_2^2\ln(\lambda_1/\lambda_3)+\lambda_3^2\ln(\lambda_2/\lambda_1)}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)}, \tag{5.14}$$
$$\Psi_3 = \frac{\ln \lambda_i/\lambda}{\lambda_i-\lambda}.$$

The method used to find (5.13) parallels that used to obtain (4.14) and (4.15) in two dimensions. Equations (2.16), (5.7), (5.10), (5.12) and (5.13) imply that *in three dimensions,* $(\ln \mathbf{V})^{\cdot}$ *is given in terms of the stretching tensor* $\mathbf{D}$ *and* $\mathbf{\Omega} = \dot{\mathbf{R}}\mathbf{R}^T$ *by*

$$(\ln \mathbf{V})^{\cdot} = \begin{cases} \dfrac{1}{(\mathrm{I\,II}-\mathrm{III})}\{\Lambda_1\mathbf{D}+\Lambda_2(\mathbf{D}\mathbf{V}+\mathbf{V}\mathbf{D})+\Lambda_3(\mathbf{D}\mathbf{V}^2+\mathbf{V}^2\mathbf{D}) \\ \quad +\Lambda_4\mathbf{V}\mathbf{D}\mathbf{V}+\Lambda_5(\mathbf{V}^2\mathbf{D}\mathbf{V}+\mathbf{V}\mathbf{D}\mathbf{V}^2)+\Lambda_6\mathbf{V}^2\mathbf{D}\mathbf{V}^2\} \\ \quad +\Psi_1(\mathbf{\Omega}\mathbf{V}^2-\mathbf{V}^2\mathbf{\Omega})+\Psi_2(\mathbf{\Omega}\mathbf{V}-\mathbf{V}\mathbf{\Omega}), & \lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1, \\ \Theta_1\mathbf{D}+\Theta_2(\mathbf{D}\mathbf{V}+\mathbf{V}\mathbf{D})+\Theta_3\mathbf{V}\mathbf{D}\mathbf{V}+\Psi_3(\mathbf{\Omega}\mathbf{V}-\mathbf{V}\mathbf{\Omega}), & \lambda_i \neq \lambda_j = \lambda_k, \\ \mathbf{D}, & \lambda_1 = \lambda_2 = \lambda_3, \end{cases} \tag{5.15}$$

where the Λ's, Θ's *and* Ψ's *are defined through* (5.8), (5.11) *and* (5.14), *respectively, and* i,j,k *is a permutation of* 1, 2, 3.

Finally, we note that eqns (5.5) and (5.6) can be used to compute $\mathbf{U}^{-1}\dot{\mathbf{U}}$ in three dimensions; when the result is incorporated into (2.15), we find

$$\mathbf{\Omega} = \mathbf{W} + \frac{1}{(\mathrm{I}\ \mathrm{II} - \mathrm{III})}[\mathrm{I}^2(\mathbf{DV}-\mathbf{VD})+\mathrm{I}(\mathbf{V}^2\mathbf{D}-\mathbf{DV}^2)+(\mathbf{VDV}^2-\mathbf{V}^2\mathbf{DV})]. \quad (5.16)\dagger$$

Substitution of (5.16) into (5.15) results in a formula for $(\ln \mathbf{V})^{\boldsymbol{\cdot}}$ in terms of $\mathbf{W}$ rather than $\mathbf{\Omega}$.

In order to compare our results with those of Hill[1], the components of (5.7), (5.10) and (5.12) are displayed in matrix form below. The stretch tensors $\mathbf{U}$ and $\mathbf{V}$ are related through

$$\mathbf{U} = \mathbf{R}^T\mathbf{V}\mathbf{R},$$

so that the way in which the rotation $\mathbf{R}$ appears in the expressions for $(\ln \mathbf{U})^{\boldsymbol{\cdot}}$ allows us to write

$$[(\ln \mathbf{U})^{\boldsymbol{\cdot}}_{ij}] = \begin{bmatrix} D_{11} & \dfrac{2\lambda_1\lambda_2 \ln(\lambda_1/\lambda_2)}{\lambda_1^2-\lambda_2^2} D_{12} & \dfrac{2\lambda_1\lambda_3 \ln(\lambda_1/\lambda_3)}{\lambda_1^2-\lambda_3^2} D_{13} \\ \bullet & D_{22} & \dfrac{2\lambda_2\lambda_3 \ln(\lambda_2/\lambda_3)}{\lambda_2^2-\lambda_3^2} D_{23} \\ \bullet & \bullet & D_{33} \end{bmatrix} \quad (5.17)$$

for $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$,

$$[(\ln \mathbf{U})^{\boldsymbol{\cdot}}_{ij}] = \begin{bmatrix} D_{11} & \dfrac{2\lambda_1\lambda \ln(\lambda_1/\lambda)}{\lambda_1^2-\lambda^2} D_{12} & \dfrac{2\lambda_1\lambda \ln(\lambda_1/\lambda)}{\lambda_1^2-\lambda^2} D_{13} \\ \bullet & D_{22} & D_{23} \\ \bullet & \bullet & D_{33} \end{bmatrix} \quad (5.18)$$

for $\lambda_1 \neq \lambda_2 = \lambda_3 \equiv \lambda$, and

$$[(\ln \mathbf{U})^{\boldsymbol{\cdot}}_{ij}] = [D_{ij}], \quad (5.19)$$

for $\lambda_1 = \lambda_2 = \lambda_3$, where the components of $(\ln \mathbf{U})^{\boldsymbol{\cdot}}$ are taken with respect to an eigenvector basis for $\mathbf{U}$, and the components of $\mathbf{D}$ are with respect to the corresponding eigenvector basis for $\mathbf{V}$ [see (2.4)]. The matrix representations of $(\ln \mathbf{U})^{\boldsymbol{\cdot}}$ for $\lambda_1 = \lambda_2 \neq \lambda_3$ and $\lambda_3 = \lambda_1 \neq \lambda_2$ are obvious variants of (5.18). Hill obtained (5.17) in [1], but did not rigorously treat the cases where the principal stretches are repeated.

6. FURTHER RESULTS

The logarithmic strain has been most successfully applied in contexts in which the principal axes of strain are fixed; there eqn (1.1) holds. However, as we have seen above, the formulas relating $\mathbf{D}$ and $\ln \mathbf{V}$ in the general case are much more complex. Here we establish a condition under which the stretching tensor and the logarithmic strain are simply related. Only the three-dimensional case is considered, but an analogous condition holds in two dimensions.

Our result will be stated in terms of the corotational and the Jaumann rates of $\ln \mathbf{V}$. The corotational derivative,

$$(\ln \mathbf{V})^\circ = (\ln \mathbf{V})^{\boldsymbol{\cdot}} + (\ln \mathbf{V})\mathbf{\Omega} - \mathbf{\Omega}(\ln \mathbf{V}),$$

† Although tangential to the topic of this paper, it is of interest that an expression for $\dot{\mathbf{R}}$ in three dimensions can easily be obtained by substitution of (5.16) into $\dot{\mathbf{R}} = \mathbf{\Omega}\mathbf{R}$ [see (2.14)].

may be combined with (2.16) to obtain

$$(\ln \mathbf{V})^{\circ} = \mathbf{R}(\ln \mathbf{U})^{\cdot}\mathbf{R}^{T}. \tag{6.1}$$

The Jaumann derivative,

$$(\ln \mathbf{V})^{*} = (\ln \mathbf{V})^{\cdot} + (\ln \mathbf{V})\mathbf{W} - \mathbf{W}(\ln \mathbf{V}),$$

when written in terms of the corotational derivative becomes

$$(\ln \mathbf{V})^{*} = (\ln \mathbf{V})^{\circ} + (\ln \mathbf{V})(\mathbf{W}-\mathbf{\Omega}) - (\mathbf{W}-\mathbf{\Omega})(\ln \mathbf{V}). \tag{6.2}$$

Recall that $\mathbf{W}-\mathbf{\Omega}$ in three dimensions is given by (5.16). Thus, by substituting (6.1), (2.7) and (5.16), and the formulas for $(\ln \mathbf{U})^{\cdot}$ derived in Section 5 into eqn (6.2), we can express $(\ln \mathbf{V})^{*}$ in terms of the distinct principal stretches, $\mathbf{V}$, and $\mathbf{D}$.

In [2] Gurtin and Spear addressed the question of whether or not a simple relation between $\ln \mathbf{V}$ and $\mathbf{D}$ exists for general deformations. They showed that on time intervals of nonzero length during which the number of principal stretches is constant

$$\mathbf{D} = (\ln \mathbf{V})^{\circ} = (\ln \mathbf{V})^{*}$$

if and only if

$$\mathbf{\Omega}_r\mathbf{U} = \mathbf{U}\mathbf{\Omega}_r. \tag{6.3}$$

Here the twirl tensor $\mathbf{\Omega}_r$ is defined through

$$\mathbf{\Omega}_r\mathbf{e}_i = \dot{\mathbf{e}}_i, \tag{6.4}$$

where the $\mathbf{e}_i$, $i = 1, 2, 3$, are the eigenvectors of $\mathbf{U}$.† By continuity, if the principal stretches are distinct at a particular time, then they are distinct in an open interval containing that time, but the intervals on which the principal values are repeated may be of zero duration (i.e. single points in time).

As our last result we prove that *the condition*

$$\mathbf{D}\mathbf{V} = \mathbf{V}\mathbf{D} \tag{6.5}$$

is necessary and sufficient for

$$\mathbf{D} = (\ln \mathbf{V})^{\circ} = (\ln \mathbf{V})^{*}. \tag{6.6}$$

First, note that (6.5) is equivalent to

$$\dot{\mathbf{U}}\mathbf{U} = \mathbf{U}\dot{\mathbf{U}}. \tag{6.7}$$

To see this observe that premultiplication of (2.12) by $\mathbf{U}$ gives

$$2\mathbf{R}^{T}\mathbf{V}\mathbf{D}\mathbf{R} = \mathbf{U}\dot{\mathbf{U}}\mathbf{U}^{-1} + \dot{\mathbf{U}},$$

and post-multiplication gives

$$2\mathbf{R}^{T}\mathbf{D}\mathbf{V}\mathbf{R} = \dot{\mathbf{U}} + \mathbf{U}^{-1}\dot{\mathbf{U}}\mathbf{U}.$$

† The restriction to intervals of nonzero length is necessary because (6.4) does not make sense at instants when the number of uniquely defined eigenvectors changes (see [2]).

When (6.5) is satisfied, the difference of these equations yields

$$\mathbf{U}^2\dot{\mathbf{U}} = \dot{\mathbf{U}}\mathbf{U}^2,$$

which is equivalent to (6.7).† Conversely, if (6.7) holds then (2.12) becomes

$$\mathbf{R}^T\mathbf{D}\mathbf{R} = \dot{\mathbf{U}}\mathbf{U}^{-1} = \mathbf{U}^{-1}\dot{\mathbf{U}}$$

so that

$$\dot{\mathbf{U}} = \mathbf{R}^T\mathbf{D}\mathbf{R}\mathbf{U} = \mathbf{R}^T\mathbf{D}\mathbf{V}\mathbf{R}$$

and

$$\dot{\mathbf{U}} = \mathbf{U}\mathbf{R}^T\mathbf{D}\mathbf{R} = \mathbf{R}^T\mathbf{V}\mathbf{D}\mathbf{R}.$$

Together these expressions for $\dot{\mathbf{U}}$ imply (6.5).

Suppose that $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$. As was previously observed, every time interval on which all three of the principal stretches are distinct will have a nonzero length, so we may apply the result of Gurtin and Spear provided that (6.7) is equivalent to (6.3) on such an interval. We now establish that this is indeed the case. Differentiation of (2.2) yields

$$\dot{\mathbf{U}} = \sum_{i=1}^{3} (\dot{\lambda}_i\mathbf{e}_i \otimes \mathbf{e}_i + \lambda_i\dot{\mathbf{e}}_i \otimes \mathbf{e}_i + \lambda_i\mathbf{e}_i \otimes \dot{\mathbf{e}}_i),$$

which, with the aid of (6.4), becomes

$$\dot{\mathbf{U}} = \sum_{i=1}^{3} (\dot{\lambda}_i\mathbf{e}_i \otimes \mathbf{e}_i + \lambda_i\boldsymbol{\Omega}_r\mathbf{e}_i \otimes \mathbf{e}_i + \lambda_i\mathbf{e}_i \otimes \boldsymbol{\Omega}_r\mathbf{e}_i). \tag{6.8}$$

Since $\boldsymbol{\Omega}_r$ is skew

$$\mathbf{e}_i \otimes \boldsymbol{\Omega}_r\mathbf{e}_i = -(\mathbf{e}_i \otimes \mathbf{e}_i)\boldsymbol{\Omega}_r$$

(no summation), so that

$$\sum_{i=1}^{3} \lambda_i\boldsymbol{\Omega}_r\mathbf{e}_i \otimes \mathbf{e}_i + \sum_{i=1}^{3} \lambda_i\mathbf{e}_i \otimes \boldsymbol{\Omega}_r\mathbf{e}_i = \boldsymbol{\Omega}_r\mathbf{U} - \mathbf{U}\boldsymbol{\Omega}_r. \tag{6.9}$$

With (6.9), (6.8) can be written as

$$\dot{\mathbf{U}} = \sum_{i=1}^{3} \dot{\lambda}_i\mathbf{e}_i \otimes \mathbf{e}_i + \boldsymbol{\Omega}_r\mathbf{U} - \mathbf{U}\boldsymbol{\Omega}_r. \tag{6.10}$$

If (6.3) holds, then by (6.10) the eigenvectors of $\mathbf{U}$ are also those of $\dot{\mathbf{U}}$ which implies (6.7). Conversely, substitution of (6.10) into (6.7) gives

$$\boldsymbol{\Omega}_r\mathbf{U}^2 = \mathbf{U}^2\boldsymbol{\Omega}_r,$$

which is equivalent to (6.3). This concludes the proof that (6.5) is necessary and sufficient for (6.6) when the principal stretches are distinct.

Suppose that $\lambda_i \neq \lambda_j = \lambda_k \equiv \lambda$, with i, j, k a permutation of 1, 2, 3. That (6.6) is valid if and only if (6.5) holds will be shown directly by use of the formula derived in Section 5. When exactly two principal stretches are equal, the appropriate expression for $(\ln \mathbf{U})^{\cdot}$ is (5.10):

$$\mathbf{R}(\ln \mathbf{U})^{\cdot}\mathbf{R}^T = \Theta_1\mathbf{D} + \Theta_2(\mathbf{V}\mathbf{D} + \mathbf{D}\mathbf{V}) + \Theta_3\mathbf{V}\mathbf{D}\mathbf{V},$$

† This follows from the fact that the characteristic spaces of $\mathbf{U}^2$ and $\mathbf{U}$ coincide (see, e.g. [3], p. 12)

where Θ_1, Θ_2 and Θ_3 are defined through (5.11). If (6.5) holds, then (5.9) can be used to rewrite (5.10) as

$$\mathbf{R}(\ln \mathbf{U})^{\cdot}\mathbf{R}^T = [\Theta_1 - \lambda_i\lambda\Theta_3]\mathbf{D} + [2\Theta_2 + (\lambda_i + \lambda)\Theta_3]\mathbf{VD}. \tag{6.11}$$

A straightforward calculation shows that

$$\begin{aligned} 2\Theta_2 &= -(\lambda_i + \lambda)\Theta_3 \\ \Theta_1 &= \lambda_i\lambda\Theta_3 + 1, \end{aligned} \tag{6.12}$$

so that (6.11) is just

$$\mathbf{R}(\ln \mathbf{U})^{\cdot}\mathbf{R}^T = \mathbf{D}. \tag{6.13}$$

By substituting (6.13) and (6.5) into (6.1), (6.2), and (5.16), we obtain (6.6).

In order to show that (6.6) implies (6.5) we use (6.1) and (5.10) to write

$$\mathbf{D} = (\ln \mathbf{V})^{\circ}$$

as

$$\mathbf{D} = \Theta_1\mathbf{D} + \Theta_2(\mathbf{VD} + \mathbf{DV}) + \Theta_3\mathbf{VDV}. \tag{6.14}$$

By expressing (6.14) in components with respect to an eigenvector basis for $\mathbf{V}$, we find that the following conditions must be satisfied:

$$1 - \Theta_1 - 2\lambda_i\Theta_2 - \lambda_i^2\Theta_3 = 0 \quad \text{or} \quad D_{ii} = 0, \tag{6.15}$$

$$1 - \Theta_1 - 2\lambda\Theta_2 - \lambda^2\Theta_3 = 0 \quad \text{or} \quad D_{jj} = D_{kk} = D_{jk} = 0, \tag{6.16}$$

$$1 - \Theta_1 - (\lambda_i + \lambda)\Theta_2 - \lambda_i\lambda\Theta_3 = 0 \quad \text{or} \quad D_{ij} = D_{ik} = 0. \tag{6.17}$$

Substitution of (6.12) into $(6.15)_1$ and $(6.16)_1$ establishes that these two equations are identically satisfied, so that (6.14) imposes no restrictions on either the diagonal terms of $\mathbf{D}$ or D_{jk}. For ease in notation let

$$\eta = \lambda_i/\lambda;$$

$\lambda_i \neq \lambda$ by assumption, so that $\eta \neq 1$. In view of (6.12) and (5.11), condition $(6.17)_1$ can be rewritten as

$$\ln \eta - \frac{\eta^2 - 1}{2\eta} = 0. \tag{6.18}$$

As the left-hand side of (6.18) is a monotonically increasing function of η, and the equation is satisfied by $\eta = 1$, (6.18) cannot hold for any $\eta \neq 1$. Therefore, $(6.17)_1$ cannot be satisfied, so that (6.14) requires $(6.17)_2$ to hold, which in turn implies that $\mathbf{D}$ and $\mathbf{V}$ commute. Thus the first equation of (6.6) implies (6.5).

It remains to consider the case where $\lambda_1 = \lambda_2 = \lambda_3$. Here $\mathbf{V}$ is isotropic and therefore commutes with all tensors; in particular (6.5) holds. Also, $(\ln \mathbf{U})^{\cdot}$ is given by (5.12), so that (6.6) follows immediately from (6.1), (6.2) and (5.16). This concludes the proof.

REFERENCES

1. R. Hill, Constitutive inequalities for isotropic elastic solids under finite strain. *Proc. R. Soc. London Ser. A* **314**, 457–472 (1970).

2. M. E. Gurtin and K. Spear, On the relationship between the logarithmic strain rate and the stretching tensor *Int. J. Solids Struct.* **19,** 437–444 (1983).
3. M. E. Gurtin, *An Introduction to Continuum Mechanics* Academic, New York (1981)
4. D. E. Carlson and A. Hoger, The derivative of a tensor-valued function of a tensor *Q Appl Math.*, to be published.
5. A. Hoger and D. E. Carlson, On the derivative of the square root of a tensor and Guo's rate theorems. *J. Elasticity* **14,** 329–336 (1984)
6. F. Sidoroff, Sur l'équation tensorielle **AX+XA = H.** *C.R. Acad. Sci. (Paris) A* **286,** 71–73 (1978)